# Math 564: Real analysis and measure theory
## Lecture 6

**Carathéodory's extension (uniqueness).** Let $\mathcal{A}$ be an algebra on a set $X$ and $\mu$ be a premeasure on $\mathcal{A}$. Then for each extension $\nu$ of $\mu$ to a measure on $\langle \mathcal{A} \rangle_\sigma$, we have $\nu \le \mu^*$. If $\mu$ is $\sigma$-finite, then $\nu = \mu^*$.

**Proof.** Since $\mu^*$ is defined as inf over covers by sets in $\mathcal{A}$, we fix a set $S \in \langle \mathcal{A} \rangle_\sigma$ and a cover $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ of $S$, and show that $\nu(S) \le \sum_{n\in\mathbb{N}} \mu(A_n)$. But this follows from ctbl subadditivity of $\nu$:

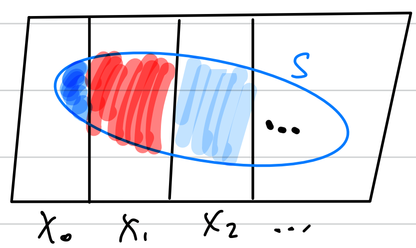
$$\nu(S) \le \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \nu(A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

This gives $\nu \le \mu^*$.

Now assume $\mu$ is $\sigma$-finite. It's actually enough to prove $\nu = \mu^*$ assuming $\mu$ is finite because given a witness to $\sigma$-finiteness, i.e. a partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ with each $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$, the fact that $\nu|_{X_n} = \mu^*|_{X_n}$ for all $n$ implies

$$\nu(S) = \nu\Big(\bigsqcup_n S \cap X_n\Big) = \sum_n \nu(S \cap X_n) = \sum_n \mu^*(S \cap X_n) = \mu^*\Big(\bigsqcup_n S \cap X_n\Big) = \mu^*(S)$$

for each $S \in \langle \mathcal{A} \rangle_\sigma$.




Thus, suppose $\mu$ is finite. We show that the function $S \mapsto \nu(S) : \langle \mathcal{A} \rangle_\sigma \to [0, \mu(X)]$ is continuous wrt the pseudo-metric $d_{\mu^*}$. Indeed, it's 1-Lipschitz:

$$|\nu(S_1) - \nu(S_2)| \le \nu(S_1 \setminus S_2) + \nu(S_2 \setminus S_1) = \nu(S_1 \Delta S_2) \le \mu^*(S_1 \Delta S_2) = d_{\mu^*}(S_1, S_2).$$

So $\nu$ and $\mu^*$ are continuous functions on $\langle \mathcal{A} \rangle_\sigma$ which coincide on a set $\mathcal{A}$ which is dense in $\langle \mathcal{A} \rangle_\sigma$ wrt $d_{\mu^*}$ because $\langle \mathcal{A} \rangle_\sigma \subseteq \overline{\mathcal{A}}^{d_{\mu^*}}$. Thus, $\nu = \mu^*$ everywhere on $\langle \mathcal{A} \rangle_\sigma$. $\square$

Thus, there are unique measures extending the Bernoulli and Lebesgue premeasures, and we call them **Bernoulli** and **Lebesgue measures**. By Bernoulli, we mean the premeasure we obtained on clopen sets on $A^{\mathbb{N}}$ for finite $A$ and any prob. measure $m$ on $A$. The corresponding Bernoulli measure is denoted by $m^{\mathbb{N}}$.

**Def.** For a metric/topological space $X$, a **Borel measure** is any measure defined on the Borel $\sigma$-algebra $\mathcal{B}(X)$.

**Examples.** Lebesgue and Bernoulli measures are examples of Borel measures. So is any Dirac measure at a point.

**Counterexample to uniqueness of extension.**

Let $\mathcal{A}$ be the algebra on $\mathbb{R}$ generated by intervals of the form $[a,b)$ for $a \leq b$. Note that $\mathcal{A}$ consists of disjoint unions of intervals of this form. Define $\mu$ on $\mathcal{A}$ by

$$\mu(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ \infty & \text{o.w.} \end{cases}$$

Then $\langle \mathcal{A} \rangle_\sigma = \mathcal{B}(\mathbb{R})$ because ctbl unions of these intervals give any open interval. The outer measure $\mu^*$ on $\mathcal{B}(\mathbb{R})$ is

$$\mu^*(B) = \begin{cases} 0 & \text{if } B = \emptyset \\ \infty & \text{o.w.} \end{cases}$$

The counting measure $\nu_c$ on $\mathcal{B}(\mathbb{R})$ is also an extension, but it's not equal to $\mu^*$:

$$1 = \nu_c(\{0\}) < \mu^*(\{0\}) = \infty.$$

Also the measure $\nu$ on $\mathcal{B}(\mathbb{R})$ defined by

$$\nu(B) := \begin{cases} 0 & \text{if } B \text{ is ctbl} \\ \infty & \text{o.w.} \end{cases}$$

is yet another extension.

We have $\nu \lneq \nu_c \leq \mu^*$ and $0 = \nu(\{0\}) < 1 = \nu_c(\{0\}) < \infty = \mu^*(\{0\})$.

# Null and measurable sets.

**Def.** Let $(X, \mathcal{B}, \mu)$ be a measure space. A set $A \subseteq X$ is called $\mu$-null if there is $B \in \mathcal{B}$ such that $A \subseteq B$ and $\mu(B) = 0$. Denote the family of all $\mu$-null sets by $\mathrm{Null}_\mu$

**Observation.** $\mu$-null sets form a $\sigma$-ideal, i.e. they are closed under subsets (downward) and under ctbl unions. In particular, if $Z$ is $\mu$-null then $\mathcal{P}(Z) \subseteq \mathrm{Null}_\mu$.

**Proof.** If the $Z_n$ are $\mu$-null, then $Z_n \subseteq \tilde{Z}_n \in \mathcal{B}$ and $\mu(\tilde{Z}_n) = 0$, so $\bigcup_{n\in\mathbb{N}} Z_n \in \bigcup_{n\in\mathbb{N}} \tilde{Z}_n$ and $\mu(\bigcup_{n\in\mathbb{N}} \tilde{Z}_n) \le \sum_{n\in\mathbb{N}} \mu(\tilde{Z}_n) = \sum_{n\in\mathbb{N}} 0 = 0$. $\square$

**Def.** For any sets $A, B \subseteq X$, write $A =_\mu B$ if $A \triangle B$ is $\mu$-null.
Call a set $A \subseteq X$ $\mu$-measurable if $A =_\mu B$ for some $B \in \mathcal{B}$. Denote by $\mathrm{Meas}_\mu$ the collection of all $\mu$-measurable sets.

**Observation.** $\mathrm{Meas}_\mu$ is $\sigma$-algebra. In fact, $\mathrm{Meas}_\mu = \langle \mathcal{B} \cup \mathrm{Null}_\mu \rangle_\sigma$.

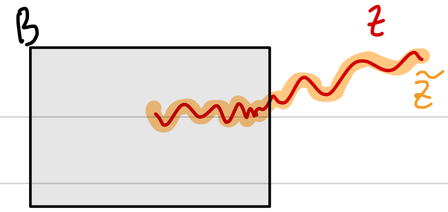
**Proof.** For complements, we have $A \triangle B$ null $\iff$ $A^c \triangle B^c$ is null because $A \triangle B = A^c \triangle B^c$.
For ctbl unions, if $A_n \triangle B_n$ is null and $B_n \in \mathcal{B}$

$\left(\bigcup_n A_n\right) \triangle \left(\bigcup_m B_m\right) \subseteq \bigcup_n (A_n \triangle B_n)$ and the latter is null. Thus $\mathrm{Meas}_\mu \supseteq \langle \mathcal{B} \cup \mathrm{Null}_\mu \rangle_\sigma$ and the other inclusion follows by the def. of $\mu$-meas. sets. $\square$

**Remark.** It is a HW exercise to show that $\mathrm{Meas}_\mu$ is what we obtain in both Carathéodory's and Tao's proofs of Carathéodory extension.

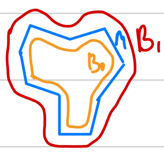
**Prop.** Let $(X, \mathcal{B}, \mu)$ be a measure space. Then:
$\{B \cup Z : B \in \mathcal{B} \text{ and } Z \text{ is } \mu\text{-null}\} = \mathrm{Meas}_\mu = \{B \setminus Z : B \in \mathcal{B} \text{ and } Z \text{ is } \mu\text{-null}\}$.

**Proof.** Since $B \cup Z$ and $B \setminus Z$ are $\mu$-meas, it's enough to show that every $\mu$-meas. set is of those two forms. Let $M$ be a $\mu$-meas. set, so $M \triangle B =: Z$ is $\mu$-null for




some $B \in \mathcal{B}$. Thus, $M = B \triangle Z$. Let $\tilde Z \supseteq Z$ be in $\mathcal{B}$ and such that $\mu(\tilde Z) = 0$. Let
$$B' := B \setminus \tilde Z \quad \text{and} \quad \tilde B := B \cup \tilde Z.$$
Then
$$B' \sqcup (B \cap (\tilde Z \setminus Z)) \sqcup (B^c \cap Z) = M = \tilde B \setminus (B \cap Z) \setminus (B^c \cap (\tilde Z \setminus Z)).$$ Check at home. □

**Cor.** For any $\mu$-meas set $M$, there are $B_0, B_1 \in \mathcal{B}$ such that
$B_1 \supseteq M \supseteq B_0$ and $\mu(B_0) = \mu(B_1)$, i.e. $B_0 \triangle M$ and $M \triangle B_1$ are $\mu$-null.




**Def.** A measure space $(X, \mathcal{B}, \mu)$ is called *complete* if $\mathcal{B} = \mathrm{Meas}_\mu$.

**Prop (completion).** Every measure $\mu$ on a measurable space $(X, \mathcal{B})$ admits a unique completion, i.e. a unique extension to a measure on $\mathrm{Meas}_\mu$.

**Proof.** Existence: Let $M$ be $\mu$-measurable, so $M = B \triangle Z$ where $B \in \mathcal{B}$ and $Z$ is $\mu$-null. Then define $\bar\mu(M) := \mu(B)$. We show that this is well-defined: if $B_0 \triangle Z_0 = M = B_1 \triangle Z_1$ with $B_i \in \mathcal{B}$ and $Z_i$ $\mu$-null, then $B_0 \triangle B_1 = (M \triangle Z_0) \triangle (M \triangle Z_1) = Z_0 \triangle Z_1 \subseteq Z_0 \cup Z_1$, so $B_0 =_\mu B_1$ hence $\mu(B_0 = B_1)$.

Uniqueness: Any extension $\nu$ satisfies $\nu(Z) = 0$ for all $Z \in \mathrm{Null}_\mu$ by monotonicity, so whenever $M = B \triangle Z$ with $B \in \mathcal{B}$ and $Z \in \mathrm{Null}_\mu$, we must have
$$\nu(M) = \nu(B) + \nu(Z) - 2 \cdot \nu(Z \cap B) = \nu(B) = \mu(B).$$ □

**Remark.** There are typically many more sets in $\mathrm{Meas}_\mu$ than in $\mathcal{B}$. For example, if $X$ is a 2nd ctbl metric/topological space, then it has at most continuum many $(= |2^{\mathbb{N}}| = |\mathbb{R}|)$ Borel sets while there are $2^{\text{continuum}} = |2^{\mathbb{R}}|$ many measurable sets if there is a continuum sized null set. For example, the standard Cantor set $C \subseteq [0,1]$ is $\lambda$-null, where $\lambda$ is Lebesgue measure, so $\mathcal{P}(C) \subseteq \mathrm{Null}_\lambda \subseteq \mathrm{Meas}_\lambda$. But $C \cong 2^{\mathbb{N}}$ so $|\mathrm{Meas}_\mu| \geq |\mathcal{P}(2^{\mathbb{N}})| = |\mathcal{P}(\mathbb{R})| = |2^{\mathbb{R}}| > |\mathbb{R}| = |\mathcal{B}(\mathbb{R})|$.